Exponential Transmission Lines as Resonators and Transformers*

RABINDRA N. GHOSE†

Summary—An attempt has been made to analyze the theory of an exponential transmission line from its complex reflection coefficient's standpoint and to indicate how the characteristics of an exponential line can be completely represented for any frequency with the help of the Smith Chart. It is shown that the optimum design parameters of an exponential transmission line which may be used as a transformer, with a frequency-sensitive load at one end, can be determined with the help of the Smith Chart and some derived equations. This paper also includes a study of the coaxial type exponential line which can be used as a series or parallel resonator. Theoretical expressions for the attenuation constant, stored energy, and Q for such types of resonator have been derived. Also indicated in this paper is the possibility of replacing the uniform-line coaxial-type resonators in many microwave and uhf wave filters by the exponential-line resonators, particularly when a large power-handling cpacity is warranted.

Introduction

In recent years the exponential transmission line has found wide application in microwave networks as a matching device suitable for matching two unequal impedances over a wide band of frequencies. A general analysis of the exponential transmission line has been made by Burrows[1] and Schelkunoff.[2] The purpose of this paper is to analyze the theory of the exponential line from its reflection coefficient's standpoint and to indicate how the characteristics of an exponential line can be represented with the help of a Smith Chart which is primarily designed for uniform transmission lines. The possibility of using the exponential transmission line in the form of a resonator is also discussed in this paper.

General Equations and Their Solution

For any transmission line system the differential equations for the voltage and current can be represented by

$$\left.\begin{aligned} \frac{dV}{dz} + Z(z)I &= 0 \\ \frac{dI}{dz} + Y(z)V &= 0 \end{aligned}\right\} \qquad (1)$$

where

V = voltage across the transmission line at z,
I = current in the transmission line at z,
Z = equivalent series impedance per unit length of the line,
Y = equivalent shunt admittance per unit length of the line.

For a nonuniform transmission line, $Z(z)$ and $Y(z)$ are functions of z; *i.e.*, they are functions of the position along the line. Eq. (1) will usually give rise to a set of second-order differential equations

$$\left.\begin{aligned} \frac{d^2V}{dz^2} - \frac{d}{dz}[\ln Z]\frac{dV}{dz} - ZYV &= 0 \\ \frac{d^2I}{dz^2} - \frac{d}{dz}[\ln Y]\frac{dI}{dz} - ZYI &= 0 \end{aligned}\right\}. \qquad (2)$$

Walker and Wax[3] have shown that these second-order differential equations can be converted into a single first-order nonlinear differential equation

$$\frac{d\rho}{dz} - 2\gamma\rho + \frac{1}{2}(1-\rho^2)\frac{d}{dz}\ln Z_0 = 0 \qquad (3)$$

where

$$\rho = \frac{V/I - Z_0}{V/I + Z_0}, \qquad Z_0 = \sqrt{Z/Y}$$

and

$$\gamma = \sqrt{ZY}.$$

For an exponential transmission line,[4]

$$Z_0 = ke^{\alpha z} \qquad (4)$$

where k is the characteristic impedance of the line at $z=0$, and α is the rate at which the characteristic impedance changes exponentially with the position z.

From (3) and (4) one obtains the differential equation for an exponential line

$$\frac{d\rho}{dz} - 2\gamma\rho + \frac{\alpha}{2}(1-\rho)^2 = 0. \qquad (5)$$

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† The Ramo-Wooldridge Corp., Los Angeles, Calif. Formerly with Radio Corp. of America, Camden, N. J.

[1] C. R. Burrows, "The exponential transmission line," *Bell Sys. Tech. J.*, vol. 17, pp. 555–573; October, 1938.

[2] S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co., Inc., New York, N. Y., p. 222.

[3] L. R. Walker and N. Wax, "Non-uniform transmission lines and reflection coefficients," *J. Appl. Phys.*, vol. 17, pp. 1043–1045; December, 1946.

[4] For an exponential line whose Z_0 decreases from $z=0$, α in (4) can be replaced by $(-\alpha)$.

If one is interested in the coaxial form of exponential line through which TEM waves propagate without cutoff, γ can be regarded as a constant function of the position z, since both α and $\beta=2\pi/\lambda$ are independent of z. This is also true for the parallel-wire transmission line and the strip transmission line carrying TEM waves.

Thus, making use of the following transformation

$$\bar{\rho}=\rho+\frac{2\gamma}{\alpha}; \qquad S^2=\left(\frac{4\gamma^2}{\alpha^2}+1\right) \tag{6}$$

one obtains from (5)

$$\int\frac{d\bar{\rho}}{\bar{\rho}^2-S^2}=\frac{\alpha}{2}\int dz+C \tag{7}$$

and

$$\rho(z)=-\left[\frac{2\gamma}{\alpha}+S\,\frac{\left(\tanh\dfrac{\alpha Sz}{2}+\tanh SC\right)}{\left(1+\tanh\dfrac{\alpha Sz}{2}\tanh SC\right)}\right] \tag{8}$$

where C is the constant of integration which has to be evaluated from the boundary condition.

Let the boundary condition be so assumed that $\rho(l)=\rho_0$, where ρ_0 is known. From (8), then,

$$\tanh SC=-\left[\frac{\left(\rho_0+\dfrac{2\gamma}{\alpha}\right)+S\tanh\dfrac{\alpha Sl}{2}}{S+\left(\rho_0+\dfrac{2\gamma}{\alpha}\right)\tanh\dfrac{\alpha Sl}{2}}\right] \tag{9}$$

and

$$\rho(z)=-\left(\frac{2\gamma}{\alpha}-\bar{\bar{Z}}_{in}\right), \tag{10}$$

where $\bar{\bar{Z}}_{in}$ is the input impedance of a uniform transmission line with normalized surge impedance of one and which is terminated at the load end, $z=l$, by an impedance

$$\bar{\bar{Z}}_l=\left(\frac{\rho_0}{S}+\frac{2\gamma}{\alpha S}\right). \tag{11}$$

As the equivalent $\bar{\bar{Z}}_{in}$ can readily be obtained from the Smith Chart when α, γ, and ρ_0 are known, the effect of the exponential-line transformer in terms of the reflection coefficient and vswr at the input end can be determined without too much laborious computation even when a frequency-sensitive load with arbitrarily varying reflection coefficient is connected at the load end. It may be remarked that while computing $\bar{\bar{Z}}_{in}$, the electrical length of the line should be considered as $\bar{\beta}l$, where

$$\bar{\beta}=\beta\sqrt{1-\frac{\alpha^2}{2\beta^2}} \tag{12}$$

for a lossless exponential line transformer.

To enable the transformer to work well beyond the cutoff frequency, it will be desirable to choose α such that

$$\alpha\ll\frac{4\pi}{\lambda}, \tag{13}$$

λ being the wavelength corresponding to the lowest frequency in the passband which has to be transmitted through the transformer. The fictitious impedance $\bar{\bar{Z}}_l$ in (11) is, in general, complex. Assuming the transformer is lossless and is operating well beyond cutoff

$$\gamma=i\beta$$

and

$$\bar{\bar{Z}}_l=\frac{\beta}{\bar{\beta}}-i\,\frac{\rho_0\alpha}{2\bar{\beta}}. \tag{14}$$

In order to make use of the method described, it is essential that $\bar{\bar{Z}}_l$ should be in the right half-plane, as otherwise one cannot make entries on the Smith Chart to compute $\bar{\bar{Z}}_{in}$. To ensure this, one can set

$$\frac{\beta}{\bar{\beta}}-\operatorname{Re}\left\{\frac{|\rho_0|}{2}\,\frac{\alpha}{\bar{\beta}}\exp\left[i\left(\theta+\frac{\pi}{2}\right)\right]\right\}\geq 0 \tag{15}$$

where θ is the phase angle of the reflection coefficient ρ_0 at the load end of the transformer. That is,

$$\beta\geq\frac{|\rho_0|}{2}\alpha\cos\left(\theta+\frac{\pi}{2}\right).$$

But $|\rho_0|\leq 1$ and the value of cos $(\theta+\pi/2)\leq 1$. Hence,

$$|\rho_0|\ \cos\left(\theta+\frac{\pi}{2}\right)\leq 1$$

and

$$\beta>\frac{\alpha}{2}.$$

But this condition has already been assumed in order to enable the transformer to operate well beyond the cutoff frequency. Hence no difficulty will be experienced in computing $\rho(0)$ according to the method described above. An equivalent circuit describing the method of computing $\rho(0)$ is shown in Fig. 1. It should be recognized that this method will be of considerable help in synthesizing the design parameters of a transformer for optimum $\rho(0)$ over a frequency band with any specific load connection.[5]

Exponential Line as a Resonator

From the preceding analysis, it appears that a standing wave can be maintained in a section of lossless exponential line when ρ_0 is chosen ± 1. This suggests the possibility of the use of the exponential line as a resonator.

[5] Schelkunoff, *op. cit.* The reflection chart shown in Fig. 7.11 can be used to determine the complex reflection coefficient when the normalized impedance of the arbitrary load is known. Similarly, the input impedance of the transformer for an arbitrary load can be determined readily from Fig. 1 above and the reflection chart.

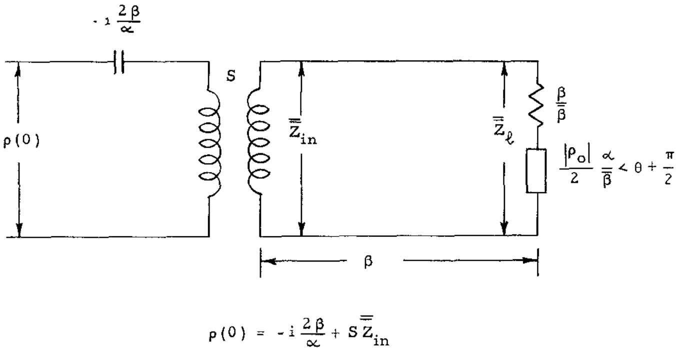

Fig. 1—Equivalent circuit of an exponential transformer.

Let it be assumed that one is interested in using a section of an exponential line as an infinite-impedance or parallel-resonance type resonator when one end of the line is short-circuited at $z=l$, as shown in Fig. 2. This is analogous to a $\lambda/4$ cavity for a uniform line

$$\rho_0 \big|_{z=l} = -1, \qquad \rho(0) = 1. \tag{16}$$

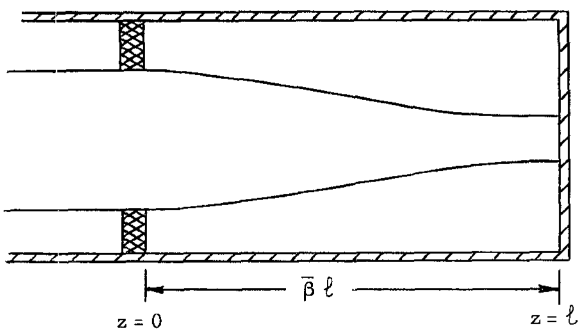

Fig. 2—Exponential resonator with Z_0 increasing with $z>0$.

From (10),

$$\frac{\alpha Sl}{2} = (2k-1)\frac{\pi}{2}$$

where k is any integer. The required length for parallel resonance is

$$l_p = \frac{\pi(2k-1)}{\sqrt{4\beta^2-\alpha^2}}$$

and

$$l_{p,\min} = \frac{\lambda}{4\left(1-\dfrac{\alpha^2}{4\beta^2}\right)^{1/2}} \tag{17}$$

where a lossless line is assumed, such that $\gamma = i\beta$. Similarly, if one is interested in using the exponential line as a series resonator,

$$\rho_0 \big|_{z=l} = -1, \qquad \rho(0) = -1, \qquad \frac{\alpha Sl}{2} = n\pi \tag{18}$$

where n is any integer.

The length of the resonator is

$$l_s = \frac{\alpha\pi n}{(4\beta^2-\alpha^2)^{1/2}}$$

and

$$l_{s,\min} = \frac{\lambda}{2\left(1-\dfrac{\alpha^2}{4\beta^2}\right)^{1/2}}. \tag{19}$$

Fig. 3 shows the required length of the resonator for series- and parallel-type resonance.

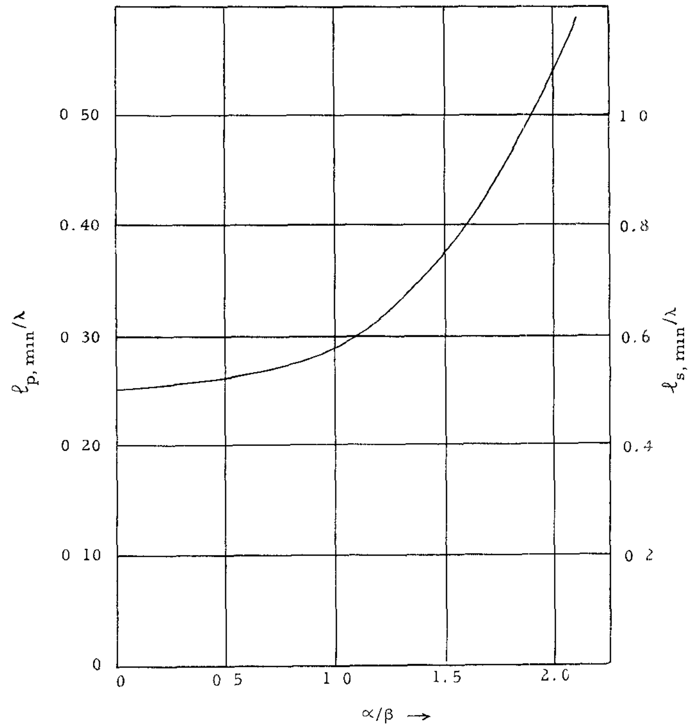

Fig. 3—Minimum resonant length of series and parallel-type exponential resonators.

Stored Energy, Wall Losses, and Q of an Exponential Resonator

From the differential equation of voltage in the exponential transmission line whose characteristic impedance increases with $z>0$, one obtains

$$V(z) = A\exp\left(\frac{\alpha z}{2} - i\bar{\beta}z\right) + B\exp\left(\frac{\alpha z}{2} + i\bar{\beta}z\right) \tag{20}$$

where A and B are arbitrary constants. When a shorting plate is placed at $z=l$,

$$V(l) = 0,$$

$$\frac{A}{B} = -e^{i2\bar{\beta}l}, \qquad \bar{\beta} = \beta\sqrt{1-\frac{\alpha^2}{4\beta^2}}$$

and

$$V(z) = -2iBe^{i\bar{\beta}l}e^{(\alpha z/2)}\sin(\bar{\beta}l - \bar{\beta}z). \tag{21}$$

Let the current at any z be represented as

$$I(\rho, z) = If(z).$$

For the TEM mode,

$$E_\rho(\rho, z) = \frac{\eta I f(z)}{2\pi\rho}$$

$$V = \int_{a(z)}^{b(z)} \frac{\eta I f(z)}{2\pi\rho} d\rho$$

and

$$E_\rho = -2iBe^{i\bar{\beta}l}e^{(\alpha z/2)}\frac{\sin(\bar{\beta}l - \bar{\beta}z)}{\left[\rho \ln \frac{b(z)}{a(z)}\right]}, \tag{22}$$

η being the intrinsic impedance of the free space. At resonance, the maximum stored electrical energy is the same as the maximum stored magnetic energy, and when the stored electrical energy is maximum, the stored magnetic energy is zero. Hence the stored energy in a section of an exponential line acting as a resonator can be obtained from its stored electrical energy alone.

Stored electrical energy

$$U = \frac{\epsilon}{2}\int_0^l \int_{a(z)}^{b(z)} \int_0^{2\pi} |E_\rho|^2 dz\rho d\rho d\phi$$

$$= M\epsilon\pi \int_0^l \int_{a(z)}^{b(z)} \frac{e^{\alpha z}\sin^2(\bar{\beta}l - \bar{\beta}z)}{\rho^2\left(\ln\frac{b(z)}{a(z)}\right)^2}\rho d\rho dz \tag{23}$$

where

$$M = 2|B|.$$

From the assumed variation of the characteristic impedance in an exponential line,

$$Z_0(z) = Z_0(0)e^{\alpha z}$$

$$= 60 \ln \frac{b(z)}{a(z)}. \tag{24}$$

Substituting the results of (24) in (23), one obtains

$$U = \frac{30\epsilon\pi M^2}{Z_0(0)}\left(l - \frac{\sin 2\bar{\beta}l}{2\bar{\beta}}\right). \tag{25}$$

For both series and parallel resonance, $\sin 2\bar{\beta}l = 0$,

$$U = \frac{30\epsilon\pi M^2 l}{Z_0(0)}$$

where ϵ is the dielectric constant of the medium inside the transformer. Fig. 4 shows a comparison of the stored energy in uniform and exponential type resonators.

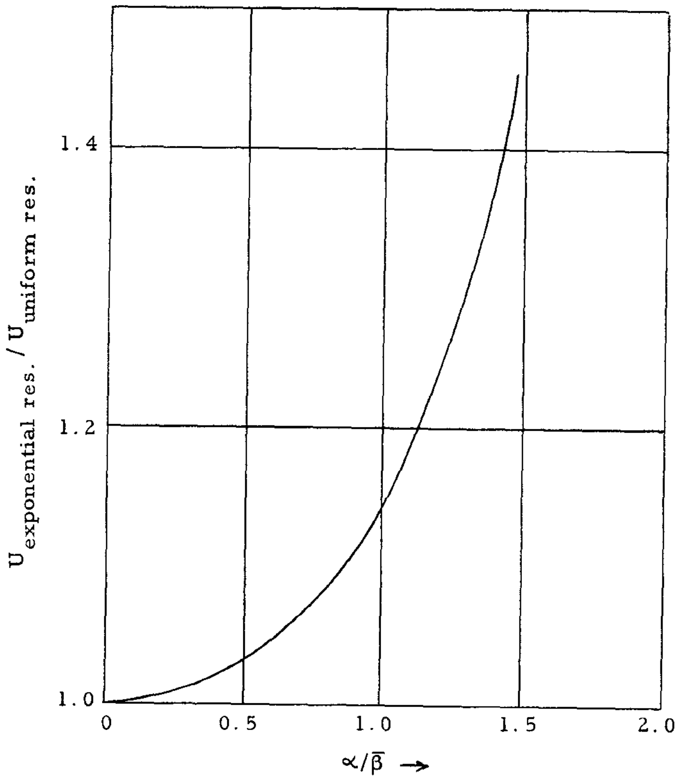

Fig. 4—Relative variation of stored energy in an exponential resonator.

The wall losses consist of the losses resulting from the tangential component of the magnetic field at the lateral surface and at the end plates of the resonator. These can be determined from the surface integral of the tangential H_ϕ over the entire surface.

For parallel resonance, analogous to the $\lambda/4$ type resonance in a uniform line resonator, the wall loss W becomes

$$W = \frac{R_s\pi M^2}{b\eta^2}\frac{3600}{Z_0^2(0)}\left\{\frac{Z_0(0)b}{60} + \frac{1}{2\alpha}\left\{\frac{\left(1 - e^{-\alpha l} + \frac{\alpha^2}{2\bar{\beta}^2}\right)}{1 + \frac{\alpha^2}{4\bar{\beta}^2}} + \sum_{m=0}^{\infty}\left(\frac{Z_0(0)}{60}\right)^m \frac{1}{m!}\frac{1}{(m-1)}\left[\frac{e^{\alpha(m-1)l} - 1 - \frac{\alpha^2(m-1)^2}{2\bar{\beta}^2}}{1 + \frac{\alpha^2(m-1)^2}{4\bar{\beta}^2}}\right]\right\}\right\} \tag{26}$$

where R_s is the real part of the surface impedance of the metal forming the resonator.

If $Z_0(0)$ be chosen 60 ohms and

$$\alpha \rightarrow 0$$

then the Q of the resonator becomes

$$\frac{240\pi^2}{R_s}\left(8+\frac{\lambda}{a}+\frac{\lambda}{b}\right)^{-1}.$$

It is interesting to note that, when

$$\alpha \rightarrow 0$$

the exponential-line resonator becomes a uniform-line resonator and the Q of such resonator[6] is

$$\frac{240\pi^2}{R_s}\left(8+\frac{\lambda}{a}+\frac{\lambda}{b}\right)^{-1}.$$

The characteristic impedance of the lines is assumed to be the same in both cases.

The Q can be determined alternatively from the input reflection coefficient already derived. But the evaluation of Q from the field integrals reveals the characteristics of the resonator from the energy consideration.

Fig. 5 shows a comparison of Q for uniform-line and exponential-line resonators for different values of α/β, when $Z_0(0)=30$ ohms.

Conclusion

An attempt has been made in this paper to describe a method by which the reflection coefficient of an exponential-line transformer can be determined readily, from the Smith Chart, particularly when a frequency-sensitive load or a transmission line whose input impedance changes with frequency is terminated at the load end. Analyses of this nature may be helpful in obtaining the optimum design parameters for the transformer for any specific load. Also discussed is the possibility of using the exponential-line resonator to indicate how, for some range of $Z_0(0)$, the Q of an exponential resonator can be increased greatly in excess of what would be expected in a uniform-line resonator for the same type of resonance. Similar analyses can be made for other nonuniform line resonators.[7]

[6] *Ibid.*, p. 280.

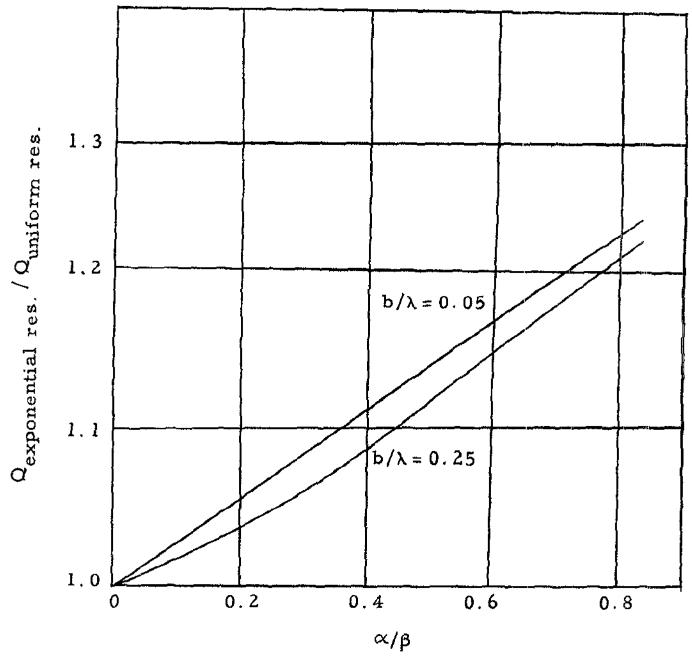

Fig. 5—Q gain in exponential resonator. Cavity length $= \dfrac{\lambda}{4\sqrt{1-\dfrac{\alpha^2}{4\beta^2}}}$.

[7] R. N. Ghose, "Synthesis of Nonuniform Line," thesis submitted in partial fulfillment of requirements for degree of electrical engineer, Univ. of Ill.; 1956.

Correction

Tore N. Anderson, author of the paper "Rectangular and Ridge Waveguide," which appeared on pages 201–209 of the October, 1956 issue of these Transactions, regrets the omission of the following reference. The illustration in Fig. 8 and the general equation for deflection of pressurized waveguide were obtained from James L. Briggs and Joseph B. Brauer, Technical Note RADC-TN-54-10, p. 3; August, 1954.